Exercise 1. 1. Let $a,b,c\in\mathbb{R}^3$. Show that

$$
\begin{aligned}
|a\times b|^2 &= |a|^2|b|^2-\langle a,b\rangle^2\\
(a\times b)\times c &= \langle a,c\rangle b-\langle b,c\rangle a.
\end{aligned}
$$

2. Let $\vec{\Phi}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^3$ be a smooth immersion. Its unit normal is the map

$$\vec{n}=\frac{\partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{\Phi}|}:\Omega\to S^2.$$

The first fundamental form is the metric given by

$$g=\vec{\Phi}^*g_{\mathbb{R}^3}=\sum_{i,j=1}^2 g_{i,j}dx_i\,dx_j,$$

where $g_{i,j}=\langle\partial_{x_i}\vec{\Phi},\partial_{x_j}\vec{\Phi}\rangle$ $(i,j=1,2)$. If $\left\{g^{i,j}\right\}_{i,j=1,2}$ is the inverse of the matrix $\{g_{i,j}\}_{1\leq i,j\leq 2}$, the Laplace-Beltrami operator is defined on $C^\infty(\Omega)$ by

$$\Delta_g f=\frac{1}{\sqrt{\det(g)}}\sum_{i,j=1}^2 g^{i,j}\partial_{x_i}\left(\sqrt{\det(g)}\partial_{x_j}f\right),\qquad f\in C^\infty(\Omega),$$

where $\det(g)=\det\{g_{i,j}\}_{1\leq i,j\leq 2}=g_{1,1}g_{2,2}-g_{1,2}^2$. Now, define the notation

$$\begin{cases}E=|\partial_{x_1}\vec{\Phi}|^2\\ G=|\partial_{x_2}\vec{\Phi}|^2\\ F=\langle\partial_{x_1}\vec{\Phi},\partial_{x_2}\vec{\Phi}\rangle\end{cases}$$

and the second fundamental form by

$$\begin{cases}\mathbb{I}_{1,1}=\left\langle\partial_{x_1}^2\vec{\Phi},\vec{n}\right\rangle\\ \mathbb{I}_{1,2}=\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\vec{n}\right\rangle\\ \mathbb{I}_{2,2}=\left\langle\partial_{x_2}^2\vec{\Phi},\vec{n}\right\rangle\end{cases}$$

Furthermore, we define the mean curvature by

$$H=\frac{1}{2}\langle\Delta_g\vec{\Phi},\vec{n}\rangle.$$

Show that

$$H=\frac{1}{2}\frac{G\,\mathbb{I}_{1,1}-2\,F\,\mathbb{I}_{1,2}+F\,\mathbb{I}_{2,2}}{EG-F^2}$$

3. Show that $\vec{\Phi}$ is a minimal immersion if and only if $H=0$.

Hint: consider variations of the form $\vec{\Phi}_t=\vec{\Phi}+t\,v\,\vec{n}$, where $v\in C_c^\infty(\Omega)$.

Exercise 2. Show that the following immersions are minimal, i.e., that their mean curvature H vanishes identically.

1. The catenoid: $\vec{\Phi}:\mathbb{R}\times S^1\to\mathbb{R}^3$, $(x,\theta)\mapsto(x,\cosh(x)\cos(\theta),\sinh(x)\sin(\theta))$.
2. The helicoid: $\vec{\Phi}:\mathbb{R}^2\to\mathbb{R}^3$, $(x,\theta)\mapsto(x\cos(y),y\sin(x),\alpha\,x)$, where $\alpha\in\mathbb{R}$.

3. The Scherk surface: $\vec{\Phi} : \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[^2 \mapsto \mathbb{R}^3, (x, y) \mapsto (x, y, \log(\cos(y)) - \log(\cos(x)))$.

Exercise 3. We say that a smooth immersion $\vec{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ has rotational symmetry if there exists $f \in C^\infty(\mathbb{R})$ such that

$$\vec{\Phi}(z, \theta) = (f(z)\cos(\theta), f(z)\sin(\theta), z).$$

Show that $\vec{\Phi}$ is a minimal surface of revolution if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ and $\mu \in \mathbb{R}$ such that

$$f(x) = \lambda \cosh\left(\lambda^{-1}(x + \mu)\right).$$